Junction Trees, Tree Width, and the Hammersley-Clifford Theorem

1 Junction Trees and Tree Width

.

Consider the equation for recursive conditioning.

$$Z(\rho, \mathcal{G}) = \sum_{y \in \mathcal{D}(Y)} Z(\rho_1[Y := y], \mathcal{G}_1) \cdots Z(\rho_k[Y := y], \mathcal{G}_k)$$

The factoring of $\mathcal{G}$ into $\mathcal{G}_1$, ..., $\mathcal{G}_k$ depends only on the set of variables $\mathrm{dom}(\rho)$ and not on the particular values that ρ assigns to those variables. This means that one can define a "factoring tree structure" with nodes of the form $n(V, \mathcal{G})$ where V is a set of variables. For technical reasons that will be clear below, we associate the computation of $Z(\rho, \mathcal{G})$ with the node $n(\mathrm{dom}(\rho) \cup \{Y\}, \mathcal{G})$ where Y is the next variable to be branched on (as in the above equation). Under this convention we have the following.

$$n(V, \mathcal{G}) \text{ has children } n(V_1, \mathcal{G}_1), \ \ldots, \ n(V_k, \mathcal{G}_k) \tag{1}$$

Here we require that no variable not in V occurs in more than one $\mathcal{G}_\rangle$ and V_i is V intersect the variables in $\mathcal{G}_\rangle$ plus the next variable in $\mathcal{G}_i$ to be cased on if $\mathcal{G}_i$ contains a variables not in V. The root of the tree is $n(\{\}, \mathcal{G})$ where $\mathcal{G}$ is the top level set of functions of the Markov random field. We also require that the branching is maximal — that each node has the maximum possible number of children allowed by these constraints. Given a variable ordering for determining which variable to branch on next, these constraints uniquely determine a tree structure with a set of variables V at each node of the tree. This is the junction tree determined by a given variable ordering. This junction tree corresponds directly to the recursive conditioning algrithm. We will call this the recursive conditioning junction tree determined by a given variable ordering.

The notion of a junction tree is more general. In general a *junction tree* for a Markov random field $\langle V, \mathcal{G} \rangle$ is a tree T each node n of which is associated with a set a variables denoted $V(n)$ with $V(n) \subseteq V$ and such that the following conditions hold.

- **Cover Property:** For each $\Gamma \in \mathcal{G}$ there exists a node n where $V(n)$ contains all variables on which Γ depends.

- **Running Intersection Property:** For any variable X, the set of nodes n with $X \in V(n)$ is a subtree of T.

The width of a junction tree is the maximum size of the set $V(n)$ over all nodes n of the tree *minus 1*. We subtract one so that the tree width of a tree is 1.

The tree width of a Markov random field is the minimum width of any junction tree for that field. It is NP hard to determine tree width. However, good heuristics exists for constructing junction trees of small width. It can be shown that there exists a variable ordering such that the recursive conditioning junction tree for that ordering has minimum width. So one can also define tree width to be the minimum over all variable orderings of the width of the recursive conditioning junction tree for that ordering. Again, good heuristics exist for selecting the next variable to branch on.

2 The Hammersley-Clifford Theorem

Comming soon.

3 Problem

Consider a greaph with nodes A_1, A_2, ..., A_n and B_1, B_2, ..., B_n and edges from A_i to B_i, from A_i to A_{i+1} and from B_i to B_{i+1}. These edges form a "ladder". Consider a Markov random field where $\mathcal{G}$ has a functrion for each edge of this graph. Give a junction tree for this MRF with width 2.